ACE 564
Spring 2006

Lecture 6

The Multiple Regression Model: Joint Hypothesis Testing

by
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Readings:

Griffiths, Hill and Judge. "Testing a Zero Null Hypothesis for all Response Coefficients," Section 10.6; "Testing a Single Linear Combination of Coefficients," Section 10.7; "Testing More than One Linear Combination of Coefficients," Section 10.8 in *Learning and Practicing Econometrics*

Kennedy. "Interval Estimation and Hypothesis Testing," Chapter 4 in *A Guide to Econometrics*

Joint Hypothesis Testing in Multiple Regression

In simple linear regression, hypothesis testing focuses on whether a parameter of the regression model is equal to a specified value

Once we move to multiple regression, hypothesis testing can assume a variety of additional forms

For example, what if we are interested in testing the same hypothesis for all of the slope parameters in a regression model?

Consider the following joint hypothesis for the Bay Area Rapid Food model,

$$H_0: \beta_2 = 0 \text{ and } \beta_3 = 0$$
$$H_1: \beta_2 \neq 0 \text{ or } \beta_3 \neq 0 \text{ or both}$$

which is usually written as,

$$H_0: \beta_2 = \beta_3 = 0$$
$$H_1: \beta_2 \neq 0, \beta_3 \neq 0, \text{or both}$$

Joint Hypothesis Tests: *F*-Test Approach

The correct approach to testing a joint hypothesis is based on a general version of the *F*-test

- Approach can accommodate any linear hypothesis or set of linear hypotheses

- Some of the joint tests also can be conducted using "simple" *t*-tests

The Basic Idea

Estimate the following full, or unrestricted, model,

$$y_t = \beta_1 + \beta_2 x_{2,t} + \beta_3 x_{3,t} + e_t$$

Now lets assume we want to test the following null and alternative hypotheses,

$$H_0 : \beta_2 = \beta_3 = 0$$
$$H_1 : \beta_2 \neq 0, \beta_3 \neq 0, \text{or both}$$

Then, impose the null hypothesis by estimating the following restricted model,

$$y_t = \beta_1 + 0x_{2,t} + 0x_{3,t} + e_t$$

or,

$$y_t = \beta_1 + e_t$$

Now, compare the *SSE* from the unrestricted multiple regression model to the *SSE* from the restricted regression model

- If these sums of squared errors **are** substantially different, then the assumption that the null hypothesis is true **has** significantly reduced the ability of the model to fit the data

  $\Rightarrow$ The data do not support the null hypothesis

- If these sums of squared errors **are not** substantially different, then the assumption that the null hypothesis is true **has not** significantly reduced the ability of the model to fit the data

  $\Rightarrow$ The data do support the null hypothesis

The General Approach

1. Specify unrestricted regression model with *K-1* independent variables

2. Estimate *K-1* variable unrestricted model and obtain the *SSE* for this unrestricted model (SSE_U)

3. Impose the *J* null hypothesis restrictions on the regression model, estimate restricted model, and obtain *SSE* for restricted model (SSE_R)

4. Form the following test statistic,

$$F = \frac{(SSE_R - SSE_U)/J}{SSE_U/(T-K)}$$

   where *J* is the number of restrictions to be tested, *T* is the number of observations and *K* is the number of parameters in the unrestricted model (Be careful here!!)

5. If the null hypothesis is true, *F* has an *F*-distribution with *J* numerator degrees of freedom and *T-K* denominator degrees of freedom

6. Compare F to the critical value from an F-distribution $F_{\alpha}(J, T\text{-}K)$, which is the critical value that leaves α percent probability in the upper tail of the F-distribution

7. If $F \geq F_{\alpha}(J, T\text{-}K)$, then the null hypothesis is rejected

Important Note

- Only equality hypotheses may be tested in this framework

- Null or alternative hypotheses may not contain "greater than or equal to" or "less than or equal to" statements

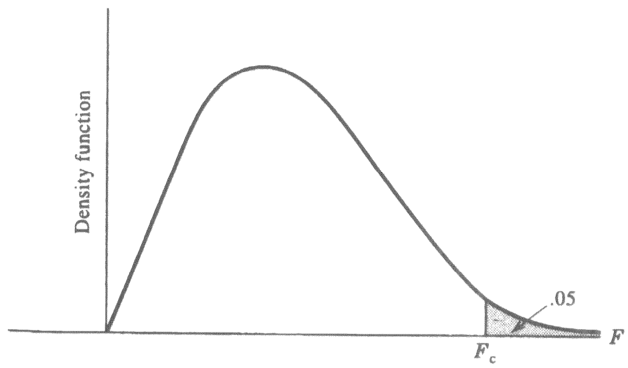


Figure 10B.1 The density function of an F random variable.

Griffiths, William E., R. Carter Hill, George G. Judge. Learning and Practicing Econometrics. John Wiley & Sons, Inc. New York. 1993.

Testing the Overall Significance of a Multiple Regression Model

Start with the following $K\text{-}1$ variable, unrestricted regression model,

$$y_t = \beta_1 + \beta_2 x_{2,t} + \beta_3 x_{3,t} + \ldots + \beta_K x_{K,t} + e_t$$

When estimated, this model will have sum of squared errors SSE_U

To examine whether we have a viable explanatory model, we set up the following null and alternative hypotheses

$$H_0 : \beta_2 = \beta_3 = \ldots = \beta_K = 0$$
$$H_1 : \text{at least one of the } \beta_k \text{ is non-zero}$$

The null hypothesis has $K-1$ parts and is a joint hypothesis

- If this null hypothesis is true, none of the explanatory variables influence y, and thus our model is of little or no value

- If the alternative hypothesis H_1 is true, then at least one of the parameters is not zero

- The alternative hypothesis does not indicate, however, which variables those might be

Since we are testing whether or not we have a viable explanatory model, the test is sometimes referred to as a test of the overall significance of the regression model

Now, impose the null hypothesis restrictions on the unrestricted regression model to obtain the following restricted model,

$$y_t = \beta_1 + 0x_{2,t} + 0x_{3,t} + \ldots + 0x_{K,t} + e_t$$

or,

$$y_t = \beta_1 + e_t$$

When estimated, this model will have sum of squared errors SSE_R

- Since $b_1 = \bar{y}$ in the estimated restricted model, the sum of squared errors for the restricted model is equal to the total sum of squares for the unrestricted model

- Hence, in this case only, where we are testing the null hypothesis that all model parameters except the intercept are zero, ***SSE_R*** equals ***SST_U***

We can then compute the test statistic F,

$$F = \frac{(SSE_R - SSE_U)/J}{SSE_U/(T-K)}$$

$$F = \frac{(SST_U - SSE_U)/J}{SSE_U/(T-K)}$$

$$F = \frac{SSR_U/J}{SSE_U/(T-K)}$$

Or, noting that $J=K-1$ in this special case,

$$F = \frac{SSR_U/(K-1)}{SSE_U/(T-K)}$$

- "Regression" F-test is automatically reported by almost all econometric packages

- Often reported in the ANOVA (Analysis of Variance) Table

Finally, compare the calculated F to the critical value from an F-distribution table, $F_\alpha(J, T-K)$, or equivalently, $F_\alpha(K-1, T-K)$

With a little more algebraic manipulation, we can derive another version of F,

$$F = \frac{R^2/(K-1)}{(1-R^2)/(T-K)}$$

Indicates "regression" F-test not only tests the overall significance of the estimated regression, but equivalently, tests the significance of R^2

Note that the "R^2 version" of F can only be derived in this special case!

Test of the Overall Significance for the Bay Area Rapid Food Regression

We want to test whether the parameters for price and advertising expenditure are both zero, against the alternative that at least one of the parameters is not zero in the following model,

$$y_t = \beta_1 + \beta_2 x_{2,t} + \beta_3 x_{3,t} + e_t$$

where y_t is total chain revenue for week t, $x_{2,t}$ is average price of chain products in week t, and $x_{3,t}$ is advertising expenditures for week t

1. Hypotheses

$$H_0: \beta_2 = \beta_3 = 0$$
$$H_1: \beta_2 \neq 0, \beta_3 \neq 0, \text{or both}$$

2. Test statistic

$$F = \frac{SSR_U / J}{SSE_U / (T-K)} = \frac{11,776.18/2}{1,805.168/(52-3)} = 159.83$$

Note that all of the information was obtained from the ANOVA table for the unrestricted regression

3. Rejection region

- Reject the null hypothesis if $F \geq F_{\alpha}(J, T\text{-}K)$

- If $\alpha = 0.05$, then $F_{\alpha}(J, T\text{-}K) = F_{0.05}(2,52\text{-}3) = 3.187$

- Reject if $F \geq 3.187$

4. Decision

- Since 159.83>3.187 we reject the null hypothesis and conclude that the estimated regression relationship is significant

- Sample evidence supports the proposition that variation in total revenue is significantly related to the variation in price and/or advertising

Sample Regression Output from Excel

SUMMARY OUTPUT

Regression Statistics	
Multiple R	**0.93117**
R Square	**0.86708**
Adjusted R Square	**0.86166**
Standard Error	**6.06961**
Observations	**52**

ANOVA

	df	*SS*	*MS*	*F*	*Significance F*
Regression	**2**	**11776.1839**	**5888.0919**	**159.8280**	**0.0000**
Residual	**49**	**1805.1684**	**36.8402**		
Total	**51**	**13581.3523**			

	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
Intercept	**104.7855**	**6.4827**	**16.1638**	**0.0000**	**91.7580**	**117.8130**
X Variable 1	**-6.6419**	**3.1912**	**-2.0813**	**0.0427**	**-13.0549**	**-0.2290**
X Variable 2	**2.9843**	**0.1669**	**17.8769**	**0.0000**	**2.6488**	**3.3198**

Relationship Between Joint and Individual Hypothesis Tests

It may appear that joint hypothesis testing using the F-test is a lot of extra (and unneeded) work

- Why not just use separate t-tests on each of the null hypotheses $H_0: \beta_2 = 0$ and $H_0: \beta_3 = 0$?

- In other words, can't we simply test (individually) a null hypothesis of zero for both parameters and obtain the correct result?

  $\Rightarrow$ The answer is NO!

We will present the explanation both mathematically and graphically

To begin, remember that testing a "zero-null" is the same as constructing a confidence interval around the estimated coefficient and asking whether zero lies inside or outside of the interval

If you need to review this equivalence, see Lecture 7 from ACE 362

The two separate CI's are,

$$\Pr[b_2 - t_{\alpha/2,T-K} se(b_2) \le \beta_2 \le b_2 + t_{\alpha/2,T-K} se(b_2)] = 1-\alpha$$

$$\Pr[b_3 - t_{\alpha/2,T-K} se(b_3) \le \beta_3 \le b_3 + t_{\alpha/2,T-K} se(b_3)] = 1-\alpha$$

We can appropriately conduct individual hypothesis tests using the above CI's

For example, if the two null hypotheses are

$H_0 : \beta_2 = 0$ and $H_0 : \beta_3 = 0$

- First compute the endpoints of each CI
- Then check to see if the hypothesized value, zero, is inside or outside the endpoints
- If inside, fail to reject relevant null
- If outside, reject relevant null

For either case, we know there is a $1-\alpha$ probability that a particular interval "covers" the null hypothesis value in repeated sampling

However, it is not also true that the probability of the intervals simultaneously "covering" $\beta_2 = 0$ and $\beta_3 = 0$ is $(1-\alpha)$

To determine the probability, lets consider the easiest case, where the sampling distributions for b_2 and b_3 are independent $[\Rightarrow \text{cov}(b_2, b_3) = 0]$

Now define the following events,

Event (A):

$$\Pr[b_2 - t_{\alpha/2,T-K}se(b_2) \le \beta_2 \le b_2 + t_{\alpha/2,T-K}se(b_2)] = 1-\alpha$$

Event (B):

$$\Pr[b_3 - t_{\alpha/2,T-K}se(b_3) \le \beta_3 \le b_3 + t_{\alpha/2,T-K}se(b_3)] = 1-\alpha$$

The multiplication rule tells us that $\Pr(A,B) = P(A)\cdot P(B)$ for independent events

So, the correct joint probability of the CIs is

$$\Pr(A,B) = P(A)\cdot P(B) = (1-\alpha)\cdot(1-\alpha)$$

In words, the probability of the confidence intervals simultaneously "covering" $\beta_2 = 0$ and $\beta_3 = 0$ is $(1-\alpha)\cdot(1-\alpha)$

Let's set $\alpha = 0.05$,

Fomby, et al

…testing a series of single hypotheses is not equivalent to testing those same hypotheses jointly. The intuitive reason is that in a joint test of several hypotheses any single hypothesis is "affected" by the information in the other hypotheses.

- This means that conflicting results can be found between separate, individual hypothesis tests, like $H_0: \beta_2 = 0$ and $H_0: \beta_3 = 0$, and a joint hypothesis test of the form $H_0: \beta_2 = \beta_3 = 0$

- We will use a graphical analysis to explore the conditions when such a conflict is most likely to arise

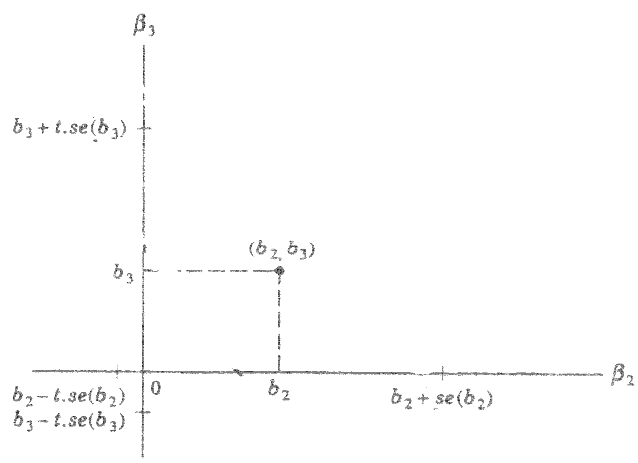
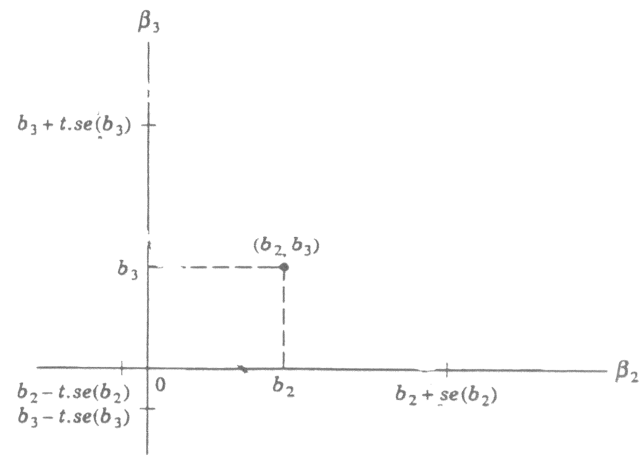


Figure 10.1 The relationship between individual and joint tests of hypotheses.

Griffiths, William E., R. Carter Hill, George G. Judge. Learning and Practicing Econometric
John Wiley & Sons, Inc. New York. 1993.



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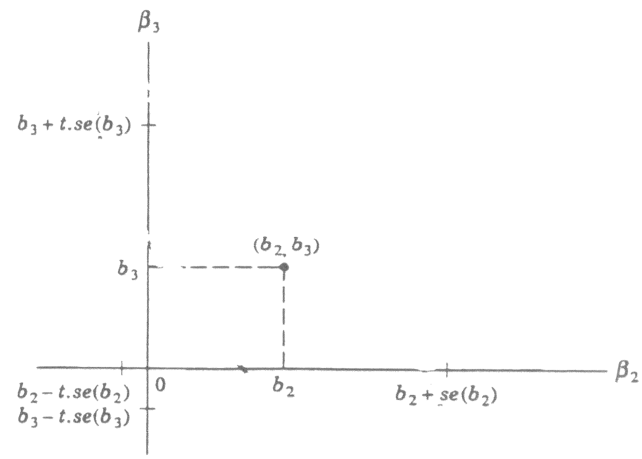

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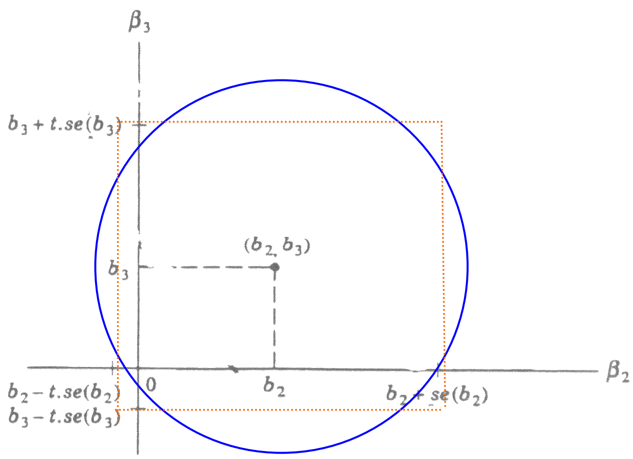

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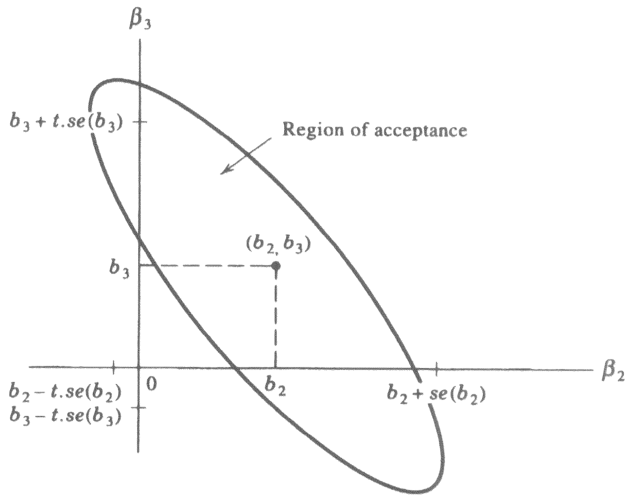

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Testing the Incremental Significance of a Group of Variables

In some situations, we may want to know the significance of a sub-group of variables in a multiple regression model

Reflects uncertainty about appropriate variables to include in the model

To demonstrate, start with the following unrestricted regression model with five independent variables,

$$y_t = \beta_1 + \beta_2 x_{2,t} + \beta_3 x_{3,t} + \beta_4 x_{4,t} + \beta_5 x_{5,t} + \beta_6 x_{6,t} + e_t$$

When estimated, this model will have sum of squared errors SSE_U

The null hypothesis of interest is,

$$H_0 : \beta_4 = \beta_5 = \beta_6 = 0$$

In other words, as a group, the last three independent variables have no linear relationship to the dependent variable

Now, impose the null hypothesis restrictions and obtain the restricted model,

$$y_t = \beta_1 + \beta_2 x_{2,t} + \beta_3 x_{3,t} + 0x_{4,t} + 0x_{5,t} + 0x_{6,t} + e_t$$

or,

$$y_t = \beta_1 + \beta_2 x_{2,t} + \beta_3 x_{3,t} + e_t$$

When estimated, this model will have sum of squared errors SSE_R

Next, compute the following test statistic,

$$F = \frac{(SSE_R - SSE_U)/J}{SSE_U/(T-K)}$$

and note that in this example, $J=3$

Finally, compare F to the critical value from an F-distribution table $F_\alpha(J, T\text{-}K)$

Test of an Extended Regression Model for Bay Area Rapid Food

So far we have proposed the following model

$$y_t = \beta_1 + \beta_2 x_{2,t} + \beta_3 x_{3,t} + e_t$$

where y_t is total chain revenue for week t, $x_{2,t}$ is average price of chain products in week t, and $x_{3,t}$ is advertising expenditures for week t

- In this model, the marginal impact of advertising is the same for all levels of advertising

- It is reasonable to expect the marginal impact of advertising to decrease as the level of advertising increases ("saturation effect" or "diminishing returns")

One way of allowing for diminishing returns to advertising is to include the squared value of advertising into the model as another explanatory variable (quadratic functional form)

So, let's consider an extended model,

$$y_t = \beta_1 + \beta_2 x_{2,t} + \beta_3 x_{3,t} + \beta_4 x_{3,t}^2 + e_t$$

This model includes a quadratic term for advertising that allows the marginal impact of advertising to vary with the level of advertising,

$$\frac{\partial y_t}{\partial x_{3,t}} = \beta_3 + 2\beta_4 x_{3,t}$$

- We expect $\beta_3 > 0$
- We expect $\beta_4 < 0$, or diminishing returns in the response to advertising

Adding another 26 weeks of data, the estimation results are

$$\hat{y}_t = 110.46 - 10.198 x_{2,t} + 3.361 x_{3,t} - 0.0268 x_{3,t}^2$$

(3.74) (1.582) (0.422) (0.0159) (s.e.)

Are the results consistent with expectations?
What is the marginal effect of advertising?

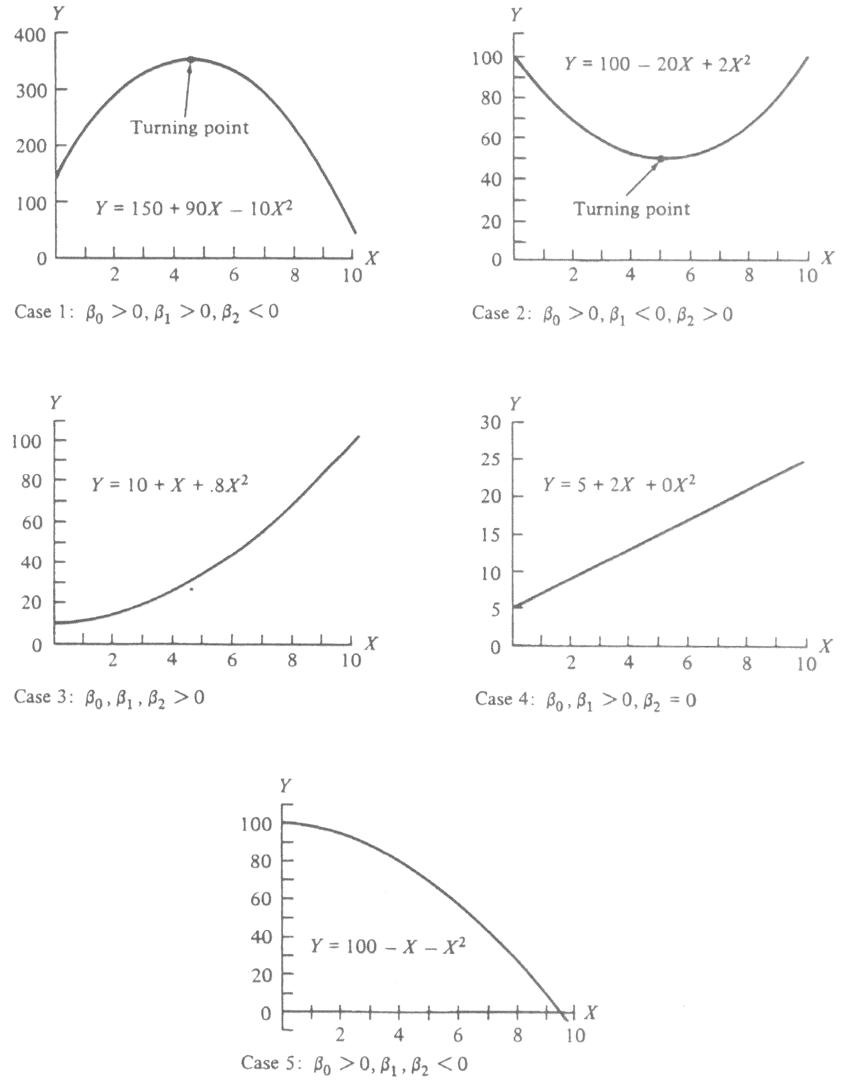

Figure 11.3 Graphs of a Quadratic Specification: $Y = \beta_0 + \beta_1 X + \beta_2 X^2$.

Johnson, A.C., Jr., M.B. Johnson and R.C. Buse. Econometrics: Basic and Applied. MacMillan Publishing Company, New York, NY 1989.

To correctly test whether the marginal impact of advertising is significant we must now test a joint hypothesis because two parameters are involved

1. Joint hypotheses

$$H_0: \beta_3 = \beta_4 = 0$$
$$H_1: \beta_3 \neq 0, \beta_4 \neq 0, \text{or both}$$

2. Test statistic

$$F = \frac{(SSE_R - SSE_U)/J}{SSE_U/(T-K)} = \frac{(20{,}907.331 - 2{,}592.301)/2}{2{,}592.304/(78-4)}$$
$$= 261.41$$

where SSE_U is obtained from the estimates for the unrestricted model,

$$y_t = \beta_1 + \beta_2 x_{2,t} + \beta_3 x_{3,t} + \beta_4 x_{3,t}^2 + e_t$$

and SSE_R is obtained from the estimates for the restricted model,

$$y_t = \beta_1 + \beta_2 x_{2,t} + e_t$$

3. Rejection region

- Reject the null hypothesis if $F \geq F_{\alpha}(J, T\text{-}K)$
- If $\alpha = 0.05$, then $F_{\alpha}(J, T\text{-}K) = F_{\alpha}(2, 78\text{-}4) = 3.120$
- Reject if $F \geq 3.120$

4. Decision

- Since 261.41>3.120 we reject the null hypothesis that both $\beta_3 = 0$ and $\beta_4 = 0$
- Conclude that at least one parameter is not zero, implying that advertising has a significant effect upon total revenue in the extended model

Testing General Linear Restrictions

In some situations, we may want to know whether a linear function of the regression parameters equals some constant

To demonstrate, again start with the following unrestricted regression model with five independent variables,

$$y_t = \beta_1 + \beta_2 x_{2,t} + \beta_3 x_{3,t} + \beta_4 x_{4,t} + \beta_5 x_{5,t} + \beta_6 x_{6,t} + e_t$$

When estimated, this model will have sum of squared errors SSE_U

The null hypothesis of interest is,

$$H_0 : \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6 = 0$$

or

$$H_0 : \beta_2 = -\beta_3 - \beta_4 - \beta_5 - \beta_6$$

- Note that sum of parameters does not have to equal zero

- Sum could be any value (e.g., 1, 9.5, -3.5, *etc*.)

Now, impose the null hypothesis restrictions and obtain the following restricted model,

$$y_t = \beta_1 + (-\beta_3 - \beta_4 - \beta_5 - \beta_6)x_{2,t} + \beta_3 x_{3,t} + \beta_4 x_{4,t} + \beta_5 x_{5,t} + \beta_6 x_{6,t} + e_t$$

$$y_t = \beta_1 + \beta_3(x_{3,t} - x_{2,t}) + \beta_4(x_{4,t} - x_{2,t}) + \beta_5(x_{5,t} - x_{2,t}) + \beta_6(x_{6,t} - x_{2,t}) + e_t$$

$$y_t = \beta_1 + \beta_3 x''_{3,t} + \beta_4 x''_{4,t} + \beta_5 x''_{5,t} + \beta_6 x''_{6,t} + e_t$$

where

$$x''_{3,t} = x_{3,t} - x_{2,t} \quad x''_{4,t} = x_{4,t} - x_{2,t}$$
$$x''_{5,t} = x_{5,t} - x_{2,t} \quad x''_{6,t} = x_{6,t} - x_{2,t}$$

When estimated, this restricted model will have sum of squared errors SSE_R

Next, compute the following test statistic,

$$F = \frac{(SSE_R - SSE_U)/J}{SSE_U/(T-K)}$$

Noting that in this example, $J=1$, compare F to the critical value from an F-distribution table $F_\alpha(J, T\text{-}K)$

Test of the Optimal Level of Advertising for Bay Area Rapid Food

Again consider the extended model,

$$y_t = \beta_1 + \beta_2 x_{2,t} + \beta_3 x_{3,t} + \beta_4 x_{3,t}^2 + e_t$$

where y_t is total chain revenue for week t, $x_{2,t}$ is average price of chain products in week t, and $x_{3,t}$ is advertising expenditures for week t

We are interested in the optimal level of advertising per week

Applying the principle that marginal revenue should equal marginal cost at the optimum,

$$\frac{\partial y_t}{\partial x_{3,t}} = \beta_3 + 2\beta_4 x_{3,t} = 1$$

Based on experience in other cities, an executive in the firm suggests that the optimum value of advertising $(x_{3,t})$ is $40,000 per week

This can be tested using the linear equality framework

1. We must now test the joint hypotheses

$$H_0: \beta_3 + 2\beta_4(40) = 1$$
$$H_1: \beta_3 + 2\beta_4(40) \neq 1$$

or,

$$H_0: \beta_3 + 80\beta_4 = 1$$
$$H_1: \beta_3 + 80\beta_4 \neq 1$$

or,

$$H_0: \beta_3 = 1 - 80\beta_4$$
$$H_1: \beta_3 \neq 1 - 80\beta_4$$

2. Test statistic

$$F = \frac{(SSE_R - SSE_U)/J}{SSE_U/(T-K)} = \frac{(2{,}594.533 - 2{,}592.301)/1}{2{,}592.304/(78-4)}$$
$$= 0.0637$$

where SSE_U is obtained from the estimates for the unrestricted model,

$$y_t = \beta_1 + \beta_2 x_{2,t} + \beta_3 x_{3,t} + \beta_4 x_{3,t}^2 + e_t$$

and SSE_R is obtained from the estimates for the restricted model,

$$y_t = \beta_1 + \beta_2 x_{2,t} + (1 - 80\beta_4)x_{3,t} + \beta_4 x_{3,t}^2 + e_t$$

$$(y_t - x_{3,t}) = \beta_1 + \beta_2 x_{2,t} + \beta_4 (x_{3,t}^2 - 80 x_{3,t}) + e_t$$

- Again, note that an additional 26 weeks of data are used in this part of the analysis

3. Rejection region

- Reject the null hypothesis if $F \geq F_\alpha(J, T\text{-}K)$

- If $\alpha = 0.05$, then $F_\alpha(J, T\text{-}K) = F_\alpha(1, 78\text{-}4) = 3.970$

- Reject if $F \geq 3.970$

4. Decision

Since $0.0637 < 3.970$ we fail to reject the null hypothesis that the optimal level of advertising per week is $40,000

Testing Equality of Parameters

In some situations, we may want to know whether regression parameters are equal for different variables

To demonstrate, start with the following unrestricted regression model with five independent variables,

$$y_t = \beta_1 + \beta_2 x_{2,t} + \beta_3 x_{3,t} + \beta_4 x_{4,t} + \beta_5 x_{5,t} + \beta_6 x_{6,t} + e_t$$

When estimated, this model will have sum of squared errors SSE_U

The null hypothesis of interest is,

$$H_0 : \beta_3 = \beta_4 \text{ and } \beta_5 = \beta_6$$

or

$$H_0 : \beta_3 - \beta_4 = 0 \text{ and } \beta_5 - \beta_6 = 0$$

Now, impose the null hypothesis restrictions and obtain the following restricted model,

$$y_t = \beta_1 + \beta_2 x_{2,t} + \beta_3 x_{3,t} + \beta_3 x_{4,t} + \beta_5 x_{5,t} + \beta_5 x_{6,t} + e_t$$

$$y_t = \beta_1 + \beta_2 x_{2,t} + \beta_3 (x_{3,t} + x_{4,t}) + \beta_5 (x_{5,t} + x_{6,t}) + e_t$$

$$y_t = \beta_1 + \beta_2 x_{2,t} + \beta_3 x''_{3,t} + \beta_5 x''_{5,t} + e_t$$

where $x''_{3,t} = x_{3,t} + x_{4,t}$ and $x''_{5,t} = x_{5,t} + x_{6,t}$

When estimated, this restricted model will have sum of squared errors SSE_R

Next, compute the following test statistic,

$$F = \frac{(SSE_R - SSE_U)/J}{SSE_U/(T-K)}$$

and note that in this example, $J=2$

Finally, compare F to the critical value from an F-distribution table $F_\alpha(J, T-K)$